Itô's calculus in physics and stochastic partial differential equations

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- Long controversy in the physical literature: Itô versus Stratonovich.
- Itô's theory to prove Stratonovich's ideas.
- Wong-Zakai's theorem for SDEs.
- Analogous results in the case of SPDEs.
- Physical relevance of Itô's correction in physics:

- wave propagation in random media,
- renormalization in quantum field theory.

Itô versus Stratonovich in SDEs

- When white noise is approximated by a smooth process this often leads to Stratonovich interpretations of stochastic integrals, at least in one dimension.

- Toy model:

$$\begin{aligned}\frac{dX^\varepsilon}{dt} &= f(X^\varepsilon)Y^\varepsilon,\\ dY^\varepsilon &= -\frac{1}{\varepsilon^2}Y^\varepsilon dt + \frac{1}{\varepsilon^2}dB.\end{aligned}$$

$Y^\varepsilon(t)$ looks like a white noise: Y^ε Gaussian, $\mathbb{E}[Y^\varepsilon(t)] = 0$, and
$\mathbb{E}[Y^\varepsilon(t)Y^\varepsilon(t')] = \frac{1}{2\varepsilon^2}\exp(-\frac{|t-t'|}{\varepsilon^2}) \to \delta(t-t')$, so the conjecture is:
$Y^\varepsilon \to \frac{dW}{dt}$ and $X^\varepsilon \to X$ (in dist.) with

$$\frac{dX}{dt} = f(X)\frac{dW}{dt}.$$

In fact, Itô's calculus shows that

$$dX = f(X)dW + \frac{1}{2}f'(X)f(X)dt,$$

which means

$$dX = f(X)\circ dW.$$

Itô versus Stratonovich in SDEs

- Wong-Zakai: under fairly general circumstances,

\- if W^{ε} denotes some "natural" smooth ε-approximation to a Brownian motion W,

\- if X^{ε} denotes the solution to the ODE

$$\frac{dX^{\varepsilon}}{dt} = h(X^{\varepsilon}) + g(X^{\varepsilon})\frac{dW^{\varepsilon}}{dt},$$

then $X^{\varepsilon} \to X$ (in dist.), the solution to the SDE

$$dX = h(X)dt + g(X) \circ dW,$$

where $\circ dW$ denotes Stratonovich integration against W.

It makes sense in the form

$$dX = h(X)dt + g(X)dW + \frac{1}{2}g'(X)g(X)dt.$$

$\hookrightarrow$ This result gives the right model for physical applications.

Itô versus Stratonovich in SDEs

Beyond one-dimensional:

- Toy model (Langevin equation):

$$\begin{aligned} m\frac{d^2X^\varepsilon}{dt^2} &= f(X^\varepsilon)Y^\varepsilon - \frac{dX^\varepsilon}{dt}, \\ dY^\varepsilon &= -\frac{1}{\varepsilon^2}Y^\varepsilon dt + \frac{1}{\varepsilon^2}dB. \end{aligned}$$

The conjecture is: $Y^\varepsilon \to \frac{dW}{dt}$ and $X^\varepsilon \to X$ with

$$m\frac{d^2X}{dt^2} = f(X)\frac{dW}{dt} - \frac{dX}{dt}.$$

Itô's calculus shows that this is correct:

$$\begin{aligned} dX &= X'dt, \\ mdX' &= f(X)dW - X'dt. \end{aligned}$$

Moreover X is smooth and the Itô and Stratonovich integrals coincide.

Itô versus Stratonovich in SDEs

Beyond one-dimensional:

- Toy model:

$$
\begin{aligned}
m_0 \varepsilon^2 \frac{d^2 X^\varepsilon}{dt^2} &= f(X^\varepsilon) Y^\varepsilon - \frac{dX^\varepsilon}{dt}, \\
dY^\varepsilon &= -\frac{1}{\varepsilon^2} Y^\varepsilon dt + \frac{1}{\varepsilon^2} dB.
\end{aligned}
$$

The conjecture is: $Y^\varepsilon \to \frac{dW}{dt}$ and $X^\varepsilon \to X$ with

$$\frac{dX}{dt} = f(X)\frac{dW}{dt}.$$

In fact, Itô's calculus shows that

$$dX = f(X)dW + \frac{1}{2(1+m_0)} f'(X) f(X) dt.$$

The integral is nor Itô (correction$=0$) neither Stratonovich (correction$=\frac{1}{2} f'(X) f(X) dt$).

Remark:

If $m_0 \varepsilon^2 \to m_0 \varepsilon$, then Itô.

If $m_0 \varepsilon^2 \to m_0 \varepsilon^3$, or $m_0 \varepsilon^4$, or $\ldots$, then Stratonovich.

Itô versus Stratonovich in SDEs

Beyond one-dimensional: Smooth approximation to white noise in one dimension leads to the Stratonovich stochastic integral.
This is not true in general, however, in the multidimensional case: an additional drift can appear in the limit.

- Toy model:

$$\begin{aligned}
\frac{dX_1^\varepsilon}{dt} &= Y_1^\varepsilon, \\
\frac{dX_2^\varepsilon}{dt} &= Y_2^\varepsilon, \\
\frac{dX_3^\varepsilon}{dt} &= (X_1^\varepsilon Y_2^\varepsilon - X_2^\varepsilon Y_1^\varepsilon), \\
dY_1^\varepsilon &= -\frac{1}{\varepsilon^2} Y_1^\varepsilon dt - \frac{\alpha}{\varepsilon^2} Y_2^\varepsilon dt + \frac{1}{\varepsilon^2} dB_1, \\
dY_2^\varepsilon &= -\frac{1}{\varepsilon^2} Y_2^\varepsilon dt + \frac{\alpha}{\varepsilon^2} Y_1^\varepsilon dt + \frac{1}{\varepsilon^2} dB_2.
\end{aligned}$$

Itô versus Stratonovich in SDEs

We conjecture

$$\begin{pmatrix} Y_1^\varepsilon \\ Y_2^\varepsilon \end{pmatrix} \to \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{dW_1}{dt} \\ \frac{dW_2}{dt} \end{pmatrix} = \frac{1}{1+\alpha^2} \begin{pmatrix} \frac{dW_1}{dt} - \alpha \frac{dW_2}{dt} \\ \alpha \frac{dW_1}{dt} + \frac{dW_2}{dt} \end{pmatrix}$$

and $(X_1^\varepsilon, X_2^\varepsilon, X_3^\varepsilon) \to (X_1, X_2, X_3)$ with

$$\begin{aligned} \frac{dX_1}{dt} &= \frac{1}{1+\alpha^2}\Big(\frac{dW_1}{dt} - \alpha \frac{dW_2}{dt}\Big), \\ \frac{dX_2}{dt} &= \frac{1}{1+\alpha^2}\Big(\frac{dW_2}{dt} + \alpha \frac{dW_1}{dt}\Big), \\ \frac{dX_3}{dt} &= \frac{1}{1+\alpha^2}\Big((\alpha X_1 - X_2)\frac{dW_1}{dt} + (\alpha X_2 + X_1)\frac{dW_2}{dt}\Big). \end{aligned}$$

Itô and Stratonovich coincide. However the result is wrong!

Correct answer:

$$dX_3 = \frac{1}{1+\alpha^2}\Big((\alpha X_1 - X_2) dW_1 + (\alpha X_2 - X_1)\ dW_2\Big) + \frac{\alpha}{1+\alpha^2} dt.$$

The drift correction is related to the Lévy area of the driving processes and the Lie brackets between the row vectors of the diffusion matrix.

Itô versus Stratonovich in SDEs

In dimension d, with r approximations of Brownian motions W_n^ε [Ikeda and Watanabe, 1989]:

$$\frac{dX_i^\varepsilon}{dt} = \sum_{n=1}^{r} \sigma_{in}(X^\varepsilon)\frac{dW_n^\varepsilon}{dt} + b_i(X^\varepsilon)$$

$$\downarrow$$

$$dX_i = \sum_{n=1}^{r} \sigma_{in}(X)dW_n + b_i(X)dt + \frac{1}{2}\sum_{n,m=1}^{r}\sum_{q=1}^{d}\big(c_{nm} + s_{nm}\big)\sigma_{qn}(X)\partial_{x_q}\sigma_{im}(X)dt$$

with the symmetric (Itô-Stratonovich) correction

$$c_{nm} = \delta_{nm}$$

and the antisymmetric (Lévy) correction

$$s_{nm} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{E}\Big[\int_0^{\varepsilon^2}\Big(W_n^\varepsilon\frac{dW_m^\varepsilon}{dt} - W_m^\varepsilon\frac{dW_n^\varepsilon}{dt}\Big)dt\Big]$$

See also [Fouque et al, 2007] for weak convergence results.

Itô versus Stratonovich - Extension to SPDEs - part I

- How to make sense of a stochastic PDE driven by noise $\frac{dW}{dt}$ *white in time* and *colored in space* of the type

$$du = \partial_x^2 u dt + H(u)dt + G(u)dW.$$

We can use martingale theory and Itô's calculus for Hilbert-space valued processes to make sense of this equation.

- Example: Itô-Schrödinger equation [Dawson and Papanicolaou, 1984]:

$$idu = \partial_x^2 u dt + u \circ dW,$$

where $\mathbb{E}[W(t,x)W(t',x')] = \min(t,t')\,\gamma(x-x')$.

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• Example: Itô-Schrödinger equation [Dawson and Papanicolaou, 1984]:

$$idu = \partial_x^2 u dt + udW - \frac{1}{2}\gamma(0)udt,$$

where $\mathbb{E}[W(t,x)W(t',x')] = \min(t,t')\,\gamma(x-x')$.

Wave propagation in random media

$$\Delta_{\vec{\boldsymbol{x}}} \hat{u}(\vec{\boldsymbol{x}}) + \frac{\omega^2}{c^2(\vec{\boldsymbol{x}})} \hat{u}(\vec{\boldsymbol{x}}) = f(\vec{\boldsymbol{x}}).$$

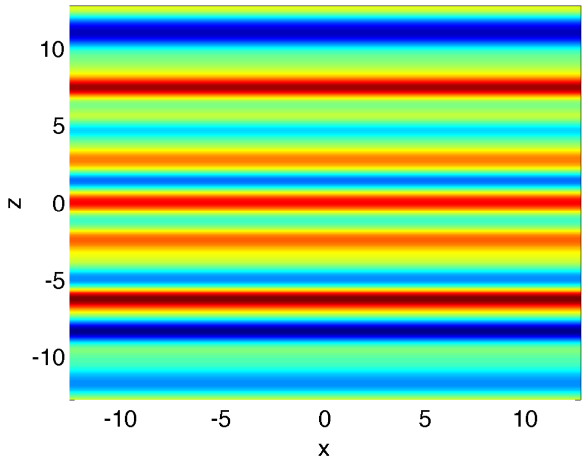
Denote $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}$.

- Randomly layered medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(z)\big)$$

c_0 is a reference speed,

$\mu(z)$ is a zero-mean random process.



- Isotropic random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(\vec{\boldsymbol{x}})\big)$$

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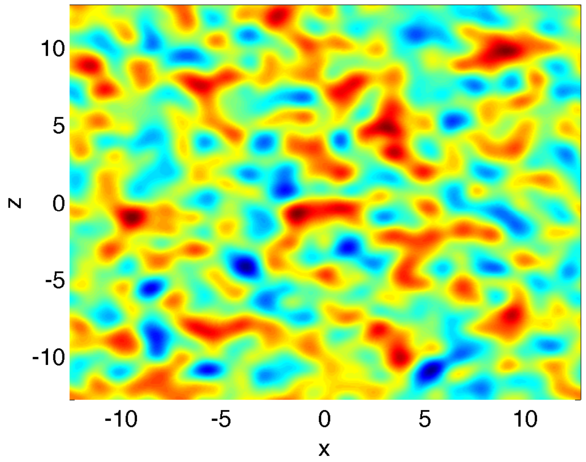

Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = \delta(z) f(\boldsymbol{x}).$$

Consider the paraxial regime:

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\Big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\Big), \qquad f(\boldsymbol{x}) \to f\Big(\frac{\boldsymbol{x}}{\varepsilon^2}\Big).$$

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = \varepsilon^4 e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\Big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\Big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\Big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\Big)\hat{\phi}^\varepsilon\right) = \delta(z) f(\boldsymbol{x}).$$

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- In the regime $\varepsilon \ll 1$, the *forward-scattering approximation* in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z),$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$, $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$, and initial conditions: $\hat{\phi}(\omega, \boldsymbol{x}, z = 0) = \frac{ic_0}{2\omega} f(\boldsymbol{x})$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

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satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\Big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\Big)\hat{\phi}^\varepsilon\right) = \delta(z) f(\boldsymbol{x}).$$

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$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} dB(\boldsymbol{x}, z) - \frac{\omega^2 \gamma(\boldsymbol{0})}{8c_0^2}\hat{\phi} dz,$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$, and initial conditions: $\hat{\phi}(\omega, \boldsymbol{x}, z = 0) = \frac{ic_0}{2\omega} f(\boldsymbol{x})$.

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Wave propagation in the random paraxial regime

- Consider the solution $\hat{\phi}(\omega, \boldsymbol{x}, z)$:

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z).$$

- By Itô's formula, the coherent wave (=mean field) satisfies

$$\partial_z \mathbb{E}[\hat{\phi}] = \frac{ic_0}{2\omega}\Delta_\perp \mathbb{E}[\hat{\phi}] - \frac{\omega^2\gamma(\mathbf{0})}{8c_0^2}\mathbb{E}[\hat{\phi}].$$

Therefore

$$\mathbb{E}\big[\hat{\phi}(\omega, \boldsymbol{x}, z)\big] = \hat{\phi}_{\rm homo}(\omega, \boldsymbol{x}, z) \exp\Big(- \frac{\gamma(\mathbf{0})\omega^2 z}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

- Exponential damping of the coherent wave.

The wave becomes incoherent.

$\Longrightarrow$ Identification of the *scattering mean free path* as the Itô-Stratonovich correction.

Wave propagation in the random paraxial regime

Consider the solution $\hat{\phi}(\omega, \boldsymbol{x}, z)$:

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z).$$

- By Itô's formula, the second-order moment

$$M\big(\omega, \boldsymbol{x}, \boldsymbol{y}, z\big) = \mathbb{E}\Big[\hat{\phi}(\omega, \boldsymbol{x}, z)\overline{\hat{\phi}(\omega, \boldsymbol{y}, z)}\Big]$$

satisfies

$$\partial_z M = \frac{ic_0}{2\omega}\big(\Delta_{\boldsymbol{x}} - \Delta_{\boldsymbol{y}}\big) M - \frac{\omega^2}{4c_0^2}\big(\gamma(\boldsymbol{0}) - \gamma(\boldsymbol{x} - \boldsymbol{y})\big) M.$$

Equivalently the Wigner transform

$$W(\omega, \boldsymbol{x}, \boldsymbol{\kappa}, z) = \int_{\mathbb{R}^2} \exp(-i\boldsymbol{\kappa} \cdot \boldsymbol{y})\mathbb{E}\left[\hat{\phi}\big(\omega, \boldsymbol{x} + \frac{\boldsymbol{y}}{2}, z\big)\overline{\hat{\phi}}\big(\omega, \boldsymbol{x} - \frac{\boldsymbol{y}}{2}, z\big)\right] d\boldsymbol{y}$$

satisfies the *radiative transport equation*

$$\partial_z W + \frac{c_0}{\omega}\boldsymbol{\kappa} \cdot \nabla_{\boldsymbol{x}} W = \frac{\omega^2}{16\pi^2 c_0^2}\int \hat{\gamma}(\boldsymbol{\kappa}')\big(W(\boldsymbol{\kappa} - \boldsymbol{\kappa}') - W(\boldsymbol{\kappa})\big) d\boldsymbol{\kappa}'.$$

The fields at nearby points are correlated.

$\Longrightarrow$ The coherent field vanishes, the wave fluctuations carry information.

Wave propagation in the random paraxial regime

• In a random medium, by Itô's formula, one can write a closed-form equation for the n-th order moment.
Depending on the statistics of the random medium, the wave fluctuations may have Gaussian statistics or not.
The wave fluctuations may have Gaussian statistics (*scintillation regime*) or not (*spot-dancing regime*) [1].

[1] J. Garnier and K. Sølna, *Comm. Part. Differ. Equat.* **39**, 626 (2014).

Itô versus Stratonovich - Extension to SPDEs - part II

• How to make sense of a stochastic PDE driven by *space-time white* noise $\frac{dW}{dt}$ (W is a L^2-valued cylindrical Wiener process) of the type

$$du = \partial_x^2 u dt + H(u)dt + G(u)dW\,?$$

• If it comes from a smooth ε-approximation of the white noise, one would expect a Stratonovich formulation.
No Stratonovich formulation for such an equation since the Itô-Stratonovich correction would be infinite.
It would be given by $\frac{1}{2}G'(u)G(u)\mathrm{Tr}(\Gamma)dt$ where $\min(t,t')\,\Gamma$ is the covariance operator of W. In the case of space-time white noise, Γ is the identity operator on L^2, which is not trace class.

Renormalization for SDEs

• Remark to Wong-Zakai for SDE: If one subtracts a suitable correction term from the random ODE, then it is possible to ensure that solutions converge to the Itô solution.

More precisely, if one considers

$$\frac{dX^\varepsilon}{dt} = h(X^\varepsilon) + g(X^\varepsilon)Y^\varepsilon - \frac{1}{2}g'(X^\varepsilon)g(X^\varepsilon),$$
$$dY^\varepsilon = -\frac{1}{\varepsilon^2}Y^\varepsilon dt + \frac{1}{\varepsilon^2}dB,$$

then $X^\varepsilon \to X$, the solution to the SDE

$$dX = h(X)dt + g(X)dW.$$

$\hookrightarrow$ Renormalization.

Renormalization and regularization for SPDEs

- Since the Itô solution is the only "natural" notion of solution available for

$$du = \partial_x^2 u dt + H(u)dt + G(u)dW,$$

for $\frac{dW}{dt}$ a space-time white noise, this suggests that if one considers approximations of the type

$$\partial_t u^\varepsilon = \partial_x^2 u^\varepsilon + H(u^\varepsilon) - C_\varepsilon G'(u^\varepsilon)G(u^\varepsilon) + G(u^\varepsilon)\xi^\varepsilon,$$

where ξ^ε is an ε-approximation to space-time white noise and C_ε is a suitable constant which diverges as $\varepsilon \to 0$, then one might expect u^ε to converge to the solution u of the SPDE, interpreted in the Itô sense.

- Almost true.

[Hairer and Pardoux, 2012] For $\rho : \mathbb{R}^2 \to \mathbb{R}$ with $\iint \rho(s,y)dsdy = 1$, consider the ε-approximation to space-time white noise:

$$\xi^\varepsilon(t,x) = \varepsilon^{-3} \int \left\langle \rho\big(\varepsilon^{-2}(t-s), \varepsilon^{-1}(x-\cdot)\big) dW(s,\cdot) \right\rangle.$$

There exist c_0, c_1, c_2 (depending on the regularization) such that, for $C_\varepsilon = c_0\varepsilon^{-1}$, we have $u^\varepsilon \to u$ where u solution of

$$du = \partial_x^2 u dt + \big[H(u) + c_1 G' G^3(u) + c_2 G'' G' G^2(u)\big] dt + G(u)dW.$$

Renormalization and regularization for SPDEs

- Result:

$$\partial_t u^\varepsilon = \partial_x^2 u^\varepsilon + H(u^\varepsilon) - c_0 \varepsilon^{-1} G'(u^\varepsilon) G(u^\varepsilon) + G(u^\varepsilon) \xi^\varepsilon$$

$$\downarrow$$

$$du = \partial_x^2 u dt + \big[H(u) + c_1 G' G^3(u) + c_2 G'' G' G^2(u) \big] dt + G(u) dW.$$

- The higher-order Itô-Stratonovich corrections involve higher powers and higher derivatives of the diffusion (volatility) term.

- Higher-order corrections were already studied for finite-dimensional systems.

$\hookrightarrow$ Corrections to Black-Scholes formula in the presence of rapidly varying stochastic volatility [Fouque et al., Derivatives in Financial Markets with Stochastic Volatility, 2000]

Renormalization and regularization for SPDEs

- Result:

$$\partial_t u^\varepsilon = \partial_x^2 u^\varepsilon + H(u^\varepsilon) - c_0 \varepsilon^{-1} G'(u^\varepsilon) G(u^\varepsilon) + G(u^\varepsilon)\xi^\varepsilon$$

$$\downarrow$$

$$du = \partial_x^2 u dt + \left[H(u) + c_1 G' G^3(u) + c_2 G'' G' G^2(u)\right] dt + G(u) dW.$$

- Conjecture (?):

$$\partial_t u^\varepsilon = \partial_x^2 u^\varepsilon + H(u^\varepsilon) - c_0 G'(u^\varepsilon) G(u^\varepsilon) + \sqrt{\varepsilon} G(u^\varepsilon)\xi^\varepsilon$$

$$\downarrow$$

$$\partial_t u = \partial_x^2 u + H(u),$$

or equivalently

$$\partial_t u^\varepsilon = \partial_x^2 u^\varepsilon + H(u^\varepsilon) + \sqrt{\varepsilon} G(u^\varepsilon)\xi^\varepsilon$$

$$\downarrow$$

$$\partial_t u = \partial_x^2 u + H(u) + c_0 G'(u) G(u).$$

Application to Allen-Cahn equation

- Consider

$$d\Phi = \Delta\Phi dt + (\Phi - \Phi^3)dt + \sigma dW, \qquad t > 0,\ \boldsymbol{x} \in \mathbb{T}^2,$$

gradient flow of the Ginzburg-Landau free energy, in a double-well potential. Here $\frac{dW}{dt}$ is an additive space-time white noise that models thermal fluctuations.

- Question/conjecture: For any $\sigma > 0$, the solution u is zero at $t > 0$?

- Regularization + renormalization: Consider additive noise white in time and colored in space

$$W^\varepsilon(t, \boldsymbol{x}) = \varepsilon^{-2} \left\langle \rho\big(\varepsilon^{-1}(\boldsymbol{x} - \cdot)\big), W(t, \cdot)\right\rangle$$

and

$$d\Phi^\varepsilon = \Delta\Phi^\varepsilon dt + (\Phi^\varepsilon - \Phi^{\varepsilon 3})dt + \sigma^\varepsilon dW^\varepsilon.$$

[Hairer, Ryzer, and Weber, 2012] If $\sigma^\varepsilon |\log \varepsilon| \to \lambda^2$, then $\Phi^\varepsilon \to \Phi$ solution of

$$\partial_t \Phi = \Delta\Phi + (\Phi - \Phi^3) - \frac{3}{8\pi}\lambda^2\Phi.$$

Conclusions

- Itô-Stratonovich correction has physical meaning.

- Only a few results available for SPDEs (recent progress by Lyons' theory on rough paths and Hairer's theory on regularity structures).

- A lot of open questions:
- Hyperbolic or dispersive problems?
- Non-Gaussian noise?
- Universal regularity structure?
- Systematic way of choosing renormalization procedure?